LITERATURE CITED

1. V. V. Bobkov, Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk, No. 4, 27-35 (1967).
2. A. A. Samarskii, Introduction to Numerical Methods [in Russian], Moscow (1982).
3. R. S. Guter and P. T. Reznikovskii, Programming and Computational Mathematics [in Russian], Vol. 2, Moscow (1971).
4. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Nat. Bureau of Standards, AMS 55, Washington (1964).
5. A. A. Samarskii, Theory of Difference Schemes [in Russian], Moscow (1979).
6. M. Yu. Shashkov, Zh. Vychisl. Mat. Mat. Fiz., 22, No. 5, 1149-1156 (1982).
7. V. K. Saul'ev, Differents. Uravn., 18, No. 12, 2190-2195 (1982).
8. V. I. Gladkovskii and V. G. Karolinskii, "Solution of a quasilinear heat conduction equation by the method of splitting with the aid of explicit asymmetric difference schemes," Deposited in VINITI, 7/5/84, No. 4739-84.
9. V. I. Gladkovskii and V. G. Karolinskii, Inzh.-Fiz. Zh., 49, No. 2, 308-314 (1985).
10. V. V. Bobkov and V. N. Shalima, Vestnik Bessarb. Gosud. Univ., Ser. I., Fiz., Mat. Mekh., No. 2, 78-80 (1979).
11. Mathematical Security, ES Computer, Inst. Mat., Akad. Nauk BSSR, No. 28, Minsk (1981).

METHOD OF INVERSE DYNAMICAL SYSTEMS FOR THE RECONSTRUCTION OF INTERNAL SOURCES AND BOUNDARY CONDITIONS IN HEAT TRANSFER

P. M. Kolesnikov, V. T. Borukhov, and L. E. Borisevich

UDC 517.958+517.977

A method for the inversion of linear dynamical systems is described; it can be used to investigate several inverse problems in the reconstruction of boundary conditions or internal sources in linear transfer equations.

The inversion of a dynamical system (DS) involves the reconstruction of unknown input signals of the system from the results of measurements of the values of certain operators defined on the instantaneous states of the DS. In the theory of energy, momentum, and mass transfer the unknown signals can be both internal and external relative to the investigated effect: time-varying amplitudes of heat and mass sources and sinks; boundary transfer conditions, e.g., boundary temperatures, boundary heat inputs, time-varying contact resistances, etc. Instrumental inverse problems, whose objective is the reconstruction of a true signal from instrument readings [1], also belongs to the class of problems of reconstruction of DS inputs.

In the linear approximation an abstract mathematical model for a broad class of transfer processes exists in the form of a differential-operator system of equations

$$\frac{\partial w}{\partial t} = Lw + Bu(t), \quad w(0) = w_0, \tag{1}$$

$$lw = 0, \tag{2}$$

which is specified in a Hilbert space H. The element w_0 of H is the initial state of the process; $w:[0, \infty] \to H$ is the transfer potential; $Bu(\cdot)$ is the source function; l is a linear operator characterizing the boundary conditions; $B:U \to H$ and $L:H \to H$ are linear operators; U is the space of values of the function $u(\cdot)$. The specific choice of the operators L, l and the space H depends on the specific details of the transfer potential, e.g., whether it is in the form of a temperature field or an electromagnetic field, and also on the characteristics of the medium, the geometry of the system, and the boundary conditions. A natural constraint identifying the given class of systems of the form (1), (2) is the fact that the restriction A of the operator L onto the set of solutions of the equation $lw = 0$ is the generating operator of a semigroup e^{At}, which is strongly continuous at zero [2] (or, in other terminology,

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 55, No. 2, pp. 304-311, August, 1988. Original article submitted March 17, 1987.

0022-0841/88/5502-0934$12.50

a semigroup of class C_0). This constraint ensures that the initial/boundary-value problem (1), (2) will be well-posed in the sense of Hadamard and, as a rule, holds for mathematical transfer models. A solution of the system of equations (1), (2) can also be written with the aid of the semigroup e^{At}:

$$w(t) = e^{At} w_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau. \tag{3}$$

We consider the inverse problem of reconstructing the source function $u(\cdot)$ from the additional information $y(\cdot)$ defined by the relation

$$y(t) = Rw, \tag{4}$$

where R is a linear operator acting from the space H into the space Y of values of the function $y(\cdot)$. According to Eqs. (3) and (4), the correspondence between $u(\cdot)$ and $y(\cdot)$ is specified by the equation

$$y(t) = Re^{At} w_0 + R \int_0^t e^{A(t-s)} Bu(s)\, ds.$$

Thus, the inverse problem of reconstructing the source function u(t) can be reduced to the solution of a Volterra operator equation of the first kind. Equations of this type have been investigated in the celebrated work of Lavrent'ev, Romanov, Bukhgeim, et al. (see [3-5] and the literature cited therein) in connection with inverse problems of integral geometry.

We consider an alternative mode of investigation of source-reconstruction inverse problems, which involves the method of inversion of linear DS's [6-9]. The limitations of the method are associated with the requirement of finiteness of the degree k of ill-posedness of the given class of inverse problems. This means that if the output data $y(\cdot)$ are analyzed in the space of operator functions absolutely continuous together with their derivatives of order up to and including $k - 1$, and the input data $u(\cdot)$ are analyzed in the space of summable operator functions, then the inverse problem is well-posed in the sense of Hadamard. Several well-known applied inverse problems, in particular the problems of reconstructing heat fluxes on the surface of a body from measurements of the temperature at interior points of the body, have an infinite degree of ill-posedness. It should be noted, on the other hand, that natural regularization techniques (e.g., differential—difference approximation of the direct problem) transform the infinite degree of ill-posedness of the inverse problem into a finite degree. In this article we ignore this possibility and proceed to investigate the interaction between the techniques of natural regularization and inversion of DS's.

From the systems point of view [10, 11], the set of equations (1), (2), (4) describes a distributed-parameter DS (which we denote for convenience by the symbol Ω), for which $u(\cdot)$ and $y(\cdot)$ are input and output signals, respectively, and $w(\cdot)$ is the state function of the DS. In the elementary case where the operator RB is inverted, the inverse of the DS Ω^{-1} has the form [6]

$$\Omega^{-1}: \begin{cases} \dfrac{\partial w}{\partial t} = (L - B(RB)^{-1}RL)\, w + B\,(RB)^{-1} \dfrac{\partial y}{\partial t}, \quad w(0) = w_0, \\ lw = 0, \\ u(t) = -(RB)^{-1}RLw + (RB)^{-1} \dfrac{\partial y}{\partial t}. \end{cases}$$

If the restriction F of the operator $L - B(RB)^{-1}RL$ onto the set of solutions of the equation $lw = 0$ forms the generating operator of a semigroup e^{Ft} of class C_0, the solution of the inverse problem of reconstructing the function $u(\cdot)$ is written directly in terms of this semigroup:

$$u(t) = -(RB)^{-1}RLe^{Ft} w_0 - (RB)^{-1}RL \int_0^t e^{F(t-s)} B\,(RB)^{-1} \frac{\partial y(s)}{\partial s}\, ds + (RB)^{-1} \frac{\partial y}{\partial t}. \tag{5}$$

The procedure developed in [6] for the structural factorization of distributed-parameter DS's can be used to formulate the inverse system when the operator RB is not invertible, but a positive integer i exists such that the system of equations

$$RBx_0 = 0,$$
$$RABx_0 + RBx_1 = 0,$$
$$\cdots\cdots\cdots$$
$$RA^iBx_0 + RA^{i-1}Bx_1 + \cdots + RBx_{i+1} = 0$$

implies $x_0 = 0$. A structural factorization procedure for the inversion of parabolic systems is also given in [12].

The structure of the solution (5) is such that it can be used to reconstruct the input signal u(t) immediately as information about the input signal y(t) is received, i.e., in real time. The ill-posed property inherent in inverse problems is also manifested in the solution (5) and is attributable to the need for differentiation of the observed quantity y(t). A second potential source of irregularity is associated with possible instability of the inverse DS in the sense of A. M. Lyapunov. The Lyapunov stability of linear DS's is known to depend on the position of the spectrum of the system generating operator in the complex plane. A procedure for formulation of the so-called reduced inverse DS has been developed [8, 9] with allowance for the latter consideration. In the reduced inverse DS the volume of information about the initial state of the system is partially curtailed, as is the spectrum of the generating operator of the inverse DS.

The boundary condition (2) is made homogeneous in the formulation of the inverse problem of reconstructing the function u(t). From the formal point of view, the homogeneity of Eq. (2) does not sacrifice generality, because a standardizing operator [13, 14] can always be used to transfer any inhomogeneity of the boundary conditions to the source involved in Eq. (1). In formulating the inverse DS, however, it is more practical to deal with the inhomogeneous boundary conditions in order to solve the problem of reconstructing transfer boundary conditions. For example, let us consider the problem of determining the function u(·) from the system of equations

$$\Sigma: \begin{cases} \dfrac{\partial w}{\partial t} = Lw, \quad w(0) = w_0, & (6) \\ lw = u(t), & (7) \\ y(t) = Rw, & (8) \end{cases}$$

where, as before, l is the operator of boundary conditions of the DS Σ, and R is the operator of observation of the states of the DS Σ. The formal representation of the inverse DS

$$\Sigma^{-1}: \begin{cases} \dfrac{\partial w}{\partial t} = Lw, \quad w(0) = w_0, & (9) \\ Rw = y(t), & (10) \\ u(t) = lw(t) & (11) \end{cases}$$

is perfectly obvious and entails the permutation of Eqs. (7) and (8), which induces a reorientation between the input and output of the primary DS Σ.

We assume that the restriction F of the operator L onto the set of solutions of the equation Rw = 0 is the generating operator of a semigroup e^{Ft} of class C_0. From Eqs. (9)-(11) we then deduce the solution of the inverse problem

$$u(t) = le^{Ft}w_0 + l\int_0^t e^{F(t-s)}B_0 y(s)\,ds. \tag{12}$$

Here B_0 is a standardizing operator, which guarantees equivalence of the system of equations (9), (10) and the system

$$\frac{\partial w}{\partial t} = Lw + B_0 y(t), \quad w(0) = 0, \quad Rw = 0.$$

The principal difficulties of implementing this approach are encountered in the proof of the existence of the semigroup e^{Ft} and its formulation. The operator R can be a point operator, a differential or integral operator, an operator of the internal superposition type, or mixed, depending on the method of observation of the states of the transfer process. Consequently, the system of equations (9), (10) represents an initial/boundary-value problem with nonclassical boundary conditions in the general case. Problems of this kind arise in various branches of science and engineering [15-19] and, beginning with the well-known investigations of Steklov [20] and Tikhonov [21], have attracted the attention of many authors [16, 22-24]. We note the work of Feller [25] and Venttsel' [26], who studied the most general additional conditions restricting an elliptic operator to the generating operator of a contractile positivity-preserving semigroup of class C_0.

We now consider examples illustrating the application of the method of inverse DS's.

Let an unbounded flat plate be given, which is thermally irradiated from one side (x = s) and is thermally insulated on the other side. The inverse problem calls for the reconstruction of the heat flux density u(t) on the surface x = s when the temperature difference y(t) = T(s, t) − T(0, t) is measured. The corresponding inverse DS has the form

$$\Sigma^{-1}: \begin{cases} c(x)T_t = (\lambda(x)T_x)_x, \quad T(x,0) = T_0(x), & (13)\\ \lambda(0)T_x(0,t) = 0, & (14)\\ T(s,t) - T(0,t) = y(t), & (15)\\ u(t) = -\lambda(s)T_x(s,t). & (16)\end{cases}$$

It follows from [25, 26] that the semigroup e^{Ft} corresponding to the DS (13)-(16) belongs to class C_0. Consequently, the solution of the inverse problems of reconstructing the heat flux density on the surface from differential temperature measurements has the form (12). Proceeding as in [27], we can show that if the coefficients c(x) and λ(x) satisfy the conditions $c(x)-c(s-x)\equiv 0$, $\lambda(x)-\lambda(s-x)\equiv 0$, $\forall x\in[0, s]$, i.e., if the graphs of the functions c(x) and λ(x) are symmetric about the line x = s/2, the reduced inverse DS $\tilde{\Sigma}^{-1}$ is described by the system of equations

$$\tilde{\Sigma}^{-1}: \begin{cases} c(x)v_t = (\lambda(x)v_x)_x, \quad v(x,0) = \dfrac{1}{2}(T_0(x) - T_0(s-x)), & (17)\\ 2v(0,t) = y(t), \quad 2v(s,t) = y(t). & (18)\\ u(t) = 2\lambda(0)v_x(0,t) & (19)\end{cases}$$

subject to Dirichlet boundary conditions (18). According to Eq. (17), for the determination of the function u(·) it is sufficient to have the odd (with respect to the axis x = s/2) part of the initial temperature distribution in the plate. In the special case of constant coefficients c and λ the solution of the stated inverse problem can be written in the analytical form [27]

$$u(t) = -\frac{2}{cs}\frac{d}{dt}\Big(\int_0^s \Omega^+(s,\vartheta,t)v_0^+(\vartheta)\,d\vartheta + \frac{\lambda}{c}\int_0^t \Omega^+(s,s,t-\tau)y(\tau)\,d\tau\Big),$$

where

$$\Omega^+(x,\vartheta,t) = 1 + 2\sum_{k=1}^{\infty}\cos\frac{2k\pi x}{s}\cos\frac{2k\pi\vartheta}{s}\exp\left(-\left(\frac{2\varkappa\pi}{s}\right)^2\frac{\lambda}{c}t\right),$$

$$v^+(x) = \int_{s/2}^{x} v(\vartheta, 0)\,d\vartheta.$$

In the next example we consider the inverse problem of reconstructing the amplitude u(t) of a neutron source $u(t)\exp(-\varkappa\sqrt{x^2+y^2})$ within the framework of the diffusion approximation for a system modeled by the two-dimensional diffusion equation

$$C_t = D(C_{xx} + C_{yy}) - b^2C + u(t)e^{-\varkappa\sqrt{x^2+y^2}}$$

subject to homogeneous Neumann boundary conditions

$$C_x(0,y,t) = C_x(s,y,t) = C_y(x,0,t) = C_y(x,s,t) = 0$$

and the zero-valued initial condition C(x, y, 0) = 0. As an additional condition, we specify the mean concentration of neutrons

$$z(t) = \frac{1}{h^2}\iint_G C(x,y,t)\,dxdy$$

in the rectangular domain $G = \{(x, y): 0 \leqslant x \leqslant h < s,\ 0 \leqslant y \leqslant h < s\}$.

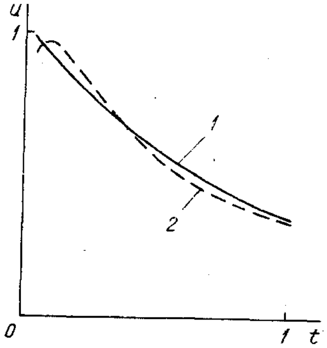

Fig. 1. Numerical modeling of the inverse problem of reconstruction of the amplitude of a neutron source: 1) primary function u(t) = exp(−t); 2) result of reconstruction of function u(t).

According to [8, 9], the reduced inverse DS is described by the system of equations

$$\begin{gathered}v_t = D(v_{xx}+v_{yy}) - b^2 v - DB(x,\ y)K^{-1}h^2\int_0^h v_x(h,\ y,\ t)\,dy + \int_0^h v_y(x,\ h,\ t)\,dx + (B(x,\ y)K^{-1}-1)\,z(t),\\ v_x(0,\ y,\ t) = v_x(s,\ y,\ t) = v_y(x,\ 0,\ t) = v_y(x,\ s,\ t) = 0,\ v(x,\ y,\ 0) = 0,\\ u(t) = -DK^{-1}h^{-2}\left(\int_0^h v_x(h,\ y,\ t)\,dy + \int_0^h v_y(x,\ h,\ t)\,dx\right) + K^{-1}\dot z(t),\end{gathered} \tag{20}$$

where

$$B(x,\ y) = e^{-\varkappa\sqrt{x^2+y^2}},\quad K = \iint_G B(x,\ y)\,dxdy.$$

We have carried out a numerical experiment in order to analyze the stability properties of the solution of the given inverse problem. The integrodifferential equation together with the boundary and initial conditions (20) was solved by a finite-difference method using an implicit scheme. The input data for the discrete model of the inverse DS were adopted in the form $\tilde z(t_i) = z(t_i)(1 + \theta_i\varepsilon)$, where θ_i are random numbers, $-1\leqslant\theta_i\leqslant 1$ and $\varepsilon = 0.1$. The noise-infiltrated function $\tilde y(t)$ was differentiated by a derivative regularization procedure [1]. Figure 1 shows the results of the numerical calculations for the following parameters of the problem: D = 1, b = 1, $\varkappa$ = 1, s = 1, h = 0.3.

In conclusion, we consider the inverse problem of reconstructing internal sources of steady radiative transfer. We restrict the discussion to a plane-parallel geometry with axial symmetry of the field. The transfer equation has the form

$$\mu\frac{\partial I_\nu(\tau,\ \mu)}{\partial\tau} + I_\nu = \frac{\Psi_\nu}{2}\int_{-1}^{1}\int_0^{2\pi} p(\mu_0)\,I_\nu(\tau,\ \mu_1)\,d\varphi d\mu_1 + B(\mu)\,u(\tau),\quad \mu_0 = \mu\mu_1 - \sqrt{(1-\mu^2)(1-\mu_1^2)}\cos\varphi. \tag{21}$$

We write the boundary conditions for Eq. (21) in the standard form

$$\begin{gathered}I_\nu(\tau,\ \mu)|_{\tau=0} = I_\nu^+(0,\ \mu),\quad 0<\mu\leqslant 1,\\ I_\nu(\tau,\ \mu)|_{\tau=\tau_0} = I_\nu^-(\tau_0,\ \mu),\quad -1\leqslant\mu\leqslant 0.\end{gathered} \tag{22}$$

The inverse problem calls for reconstruction of u(·) on the interval $0\leqslant\tau\leqslant\tau_0$ from the results of measurements of the integral field characteristic

$$y(\tau) = \int_{-1}^{1} r(\mu)\,\mu I_\nu(\tau,\ \mu)\,d\mu,\quad 0\leqslant\tau\leqslant\tau_0. \tag{23}$$

To solve the stated inverse problem, we exploit the analogy between the evolution equation of the DS and the steady transfer equation (21), based on the analogy of time and the optical thickness coordinate τ. We differentiate the equation of the inverse problem with respect to τ and use the substitution $I_\nu(\tau,\mu) = v(\tau,\ \mu) + p_+(\mu)y(\tau)$, where $v(\tau,\ \mu)$ satisfies the identity

$$\int_{-1}^{1} r(\mu)\,\mu v(\tau,\ \mu)\,d\,d\mu = 0, \quad \forall\tau \in [0,\ \tau_0],$$

and the function $p_+(\mu)$ is determined from the condition

$$\int_{-1}^{1} p_+(\mu)\, r(\mu)\,\mu d\mu = 1.$$

As a result, we obtain

$$\dot{y}(\tau) + \langle r,\ I_\nu \rangle = \langle r,\ PI_\nu \rangle + Ku(\tau),$$

where <•, •> denotes the scalar product in the Hilbert space $L_2[-1, 1]$, $K = \langle r, B\rangle$,

$$PI_\nu = \frac{\psi\nu}{2} \int_{-1}^{1}\int_{0}^{2\pi} p(\mu_0)\, I_\nu(\tau,\ \mu_1)\, d\varphi d\mu_1. \tag{24}$$

Assuming that $K \neq 0$ and carrying out simple transformations, from Eqs. (21)-(24) we deduce the realization of the inverse system

$$\mu \frac{\partial v}{\partial \tau} + v = Pv + BK^{-1} \langle r - Pr,\ v \rangle +$$

$$+ (Pp_+ - p_+ + BK^{-1} \langle r - Pr,\ p_+ \rangle)\, y(\tau) + (BK^{-1} - p_+)\dot{y}(\tau), \tag{25}$$

$$v(\tau,\ \mu)|_{\tau=0} = I_\nu^+(0,\ \mu) - p_+(\mu)\, y(0) \quad \text{for} \quad 0 \leqslant \mu \leqslant 1,$$

$$v(\tau,\ \mu)|_{\tau=\tau_0} = I_\nu^-(\tau_0,\ \mu) - p_+(\mu)\, y(\tau_0) \quad \text{for} \quad -1 \leqslant \mu < 0, \tag{26}$$

$$u(\tau) = K^{-1} \langle r - Pr,\ v \rangle + K^{-1} \langle r - Pr,\ p_+ \rangle y + K^{-1}\dot{y}(\tau). \tag{27}$$

As in the case of radiative transfer problems, the boundary-value problem (25), (26) can be solved by numerical methods. The solution of the inverse problem is expressed in terms of the solution of the boundary-value problem (25), (26) by means of Eq. (27).

NOTATION

L, B, l, linear operators; H, Hilbert space; T, temperature field; c, specific heat; λ, thermal conductivity; C, concentration of neutrons; D, diffusion coefficient; b, material parameter of reactor; $I_\nu(\tau,\ \mu)$, intensity of radiation of frequency ν at point τ in direction $\theta = \cos^{-1}\mu$; p, angular scattering function; $L_2[-1, 1]$, Hilbert space of functions summable in the square on interval $[-1, 1]$.

LITERATURE CITED

1. A. N. Tikhonov and V. S. Arsenin, Solutions of Ill-Posed Problems, Halsted Press, New York (1977).
2. E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, Am. Math. Soc., Providence, RI (1957).
3. M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskii, Some Improperly Posed Problems of Mathematical Physics, Springer-Verlag, Berlin (1967).
4. V. G. Romanov, Inverse Problems of Mathematical Physics [in Russian], Moscow (1984).
5. A. L. Bukhgeim, Volterra Equations and Inverse Problems [in Russian], Novosibirsk (1983).
6. V. T. Borukhov, Avtomat. Telemekh., No. 5, 29-36 (1982).
7. V. T. Borukhov and P. M. Kolesnikov, Izv. Akad. Nauk SSSR Tekh. Kibern., No. 3, 168-174 (1983).
8. V. T. Borukhov and P. M. Kolesnikov, Application of the Method of Inverse Dynamical Systems for the Reconstruction of Internal Heat Sources, Preprint Inst. Heat Mass Transfer Acad. Sci. B. SSR No. 11 [in Russian], ITMO AN B. SSR, Minsk (1986).
9. V. T. Borukhov and L. E. Borisevich, in: Mathematical Models, Analytical and Numerical Methods in Transfer Theory, Proc. Int. School [in Russian], Part 2, Minsk (1986), pp. 124-132 (collected scientific papers, ITMO AN B. SSR).
10. M. D. Mesarovic and Y. Takahara, General Systems Theory: Mathematical Foundations, Academic Press, New York (1975).
11. V. T. Borukhov, in: Mathematical Models of Transfer Theory in Inhomogeneous and Nonlinear Media with Phase Transitions [in Russian], Minsk (1986), pp. 46-61 (collected scientific papers, ITMO AN B. SSR).

12. S. N. Singh, IEEE Trans. Automat. Control, AC-27, No. 1, 276-279 (1982).
13. A. G. Butkovskii, Characteristics of Distributed-Parameter Systems [in Russian], Moscow (1979).
14. V. É. Lyantse and O. G. Storozh, Methods of the Theory of Unbounded Operators [in Russian], Kiev (1983).
15. A. V. Lykov and Yu. A. Mikhailov, Theory of Heat and Mass Transfer [in Russian], Moscow (1963).
16. A. A. Samarskii, Differents. Uravn., 16, No. 11, 1925-1935 (1980).
17. A. M. Nakhushev, Differents. Uravn., 19, No. 1, 86-93 (1983).
18. P. M. Kolesnikov, Methods of Transfer Theory in Nonlinear Media [in Russian], Minsk (1981).
19. P. M. Kolesnikov, Int. J. Heat Mass Transfer, 30, No. 1, 85-92 (1987).
20. V. A. Steklov, Fundamental Problems of Mathematical Physics [in Russian], Parts 1 and 2, Petrograd (1923).
21. A. N. Tikhonov, Mat. Sb., 26, No. 1, 35-56 (1950).
22. Yu. K. Lando, Differents. Uravn., 14, No. 11, 545-548 (1971).
23. L. I. Kamynin, Zh. Vychisl. Mat. Mat. Fiz., 16, No. 6, 1480-1488 (1976).
24. N. I. Yurchuk, Differents. Uravn., 22, No. 12, 2117-2126 (1986).
25. W. Feller, Ann. Math., 55, 468-519 (1952).
26. A. D. Venttsel', Teor. Veroyatn. Prim., 4, No. 2, 172-185 (1959).
27. V. T. Borukhov, Inzh.-Fiz. Zh., 47, No. 3, 469-474 (1984).